# HEATING OF THIN-WALL SHELLS OF REVOLUTION

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The temperature field is determined in thin-wall shells of revolution. The solved problems enable one to estimate the possibility as well as the quality of thermostatic control of objects by means of heat flowing along the length of the shell.

Many elements of design equipment operating under vacuum conditions are in the form of shells of revolution whose temperature is determined by heat radiation into the surrounding space and also by convective heat exchange with gas (or liquid) inside the shell, as well as by the distribution of heat sources and sinks (various units and systems releasing or absorbing heat) along the shell.

It is very difficult to determine the temperature field of a shell in the general case. However, in two limiting cases which are of practical interest it is possible to find exact solutions for some shell shapes.

If the inequality $\lambda\delta/hL^2 \ll 1$ is satisfied, one can ignore the heat flowing along the length of the shell. Moreover, if there is no convective heat exchange, the temperature field in the shell is determined by the heat flux to various portions, by shell radiation into space, and by radiative heat exchange between shell elements [2, 3].

The other limiting case $\lambda\delta/hL^2 \gg 1$ is now considered; consequently, the flow down the shell is essential insofar as one is able to linearize the problem, that is, the specific heat flux which corresponds to the shell radiation into space can be written as

$$q_1 \approx 4\sigma T_0^3 T - 3\sigma T_0^4 , \tag{1}$$

where $T_0$ is the absolute temperature of any point of the shell. It is assumed that by virtue of the flow down the mid-surface one has $(T-T_0) \ll T_0$. It is also assumed that the mid-surface of the shell is cut out by one or two planes perpendicular to the axis of revolution zone of the coordinate surface $\xi = \xi_0$ in the coordinate system ($\xi$, $\eta$, $\varphi$) in which the variables can be separated in the Laplace operator for the three-dimensional space or for a given coordinate surface either directly or by introducing an auxiliary function. In agreement with the above, one considers shells bounded by two coordinate surfaces ($\xi = \xi_1$; $\xi = \xi_2$; $\xi_1 + \xi_2 = 2\xi_0$; $\beta = \xi_2 - \xi_1 =$ const) or shells of constant thickness ($\delta = \beta H_\xi =$ const) (Fig. 1).

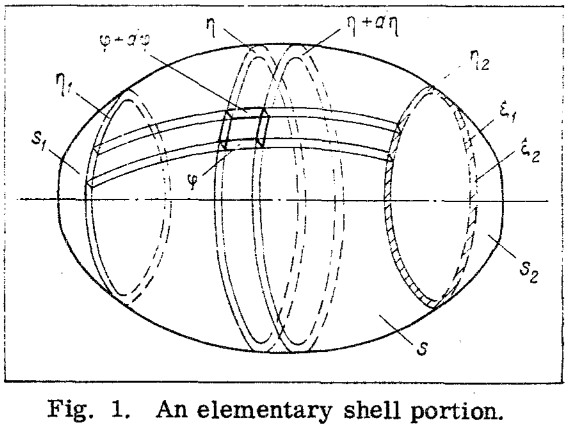


Fig. 1. An elementary shell portion.

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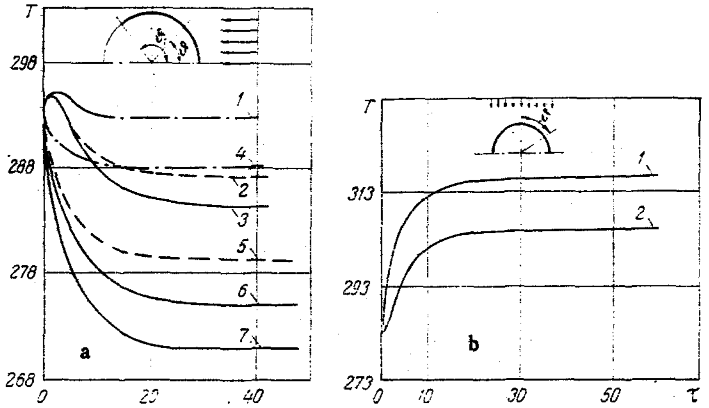


**Fig. 2.** Heating of the segments of a spherical shell which correspond to the following angles: a) $\vartheta_1 = \pi$ — solid lines; $\vartheta_1 = {}^2/_3\pi$ — dashed lines; $\vartheta_1 = \pi/2$ — dashed-dot lines during time $\tau$ (min) for $\varepsilon = 0.9$ and $A_S = 0.4$ at the shell points [curves 1, 2, 3 — $\vartheta = 0$ (front point); 4, 5, 6 — $\vartheta = \pi/2$; and 7 — $\vartheta = \pi$]; b) $\vartheta_1 = \pi$ for $\varepsilon = 0.9$ and $A_S = 0.9$ at the points $\vartheta = 0(\pi)$ (curve 1) and $\vartheta = \pi/2$ (curve 2).

If one ignores the temperature dependence on the shell thickness and uses the relation (1), the heat-conduction equation for the first case ($\beta$ = const, $\beta H_\xi \neq$ const) can be written as

$$c\rho\beta H_\xi \frac{\partial T}{\partial \tau} = \lambda\beta H_\xi \Delta_1 T + q - hT. \tag{2}$$

Here q is the specific heat flux (per unit of shell surface) with the second term on the right-hand side of the relation (1) taken into account. The operator is given by

$$\Delta_1 = \frac{1}{H_\xi H_\eta H_\varphi}\left[\frac{\partial}{\partial \eta}\left(\frac{H_\xi H_\varphi}{H_\eta}\cdot\frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{H_\xi H_\eta}{H_\varphi}\cdot\frac{\partial}{\partial \varphi}\right)\right].$$

It was assumed in the derivation of Eq. (2) that there is now radiative heat exchange between individual portions* of the shell or that it can be ignored.

The heat-conduction equation for a shell of constant thickness $\delta$ is, correspondingly,

$$c\rho\delta \frac{\partial T}{\partial \tau} = \lambda\delta\Delta T + q - hT, \tag{3}$$

where

$$\Delta = \frac{1}{H_\eta H_\varphi}\left[\frac{\partial}{\partial \eta}\left(\frac{H_\varphi}{H_\eta}\frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{H_\eta}{H_\varphi}\frac{\partial}{\partial \varphi}\right)\right].$$

The homogeneous boundary conditions of the second kind are specified (no heat dissipation) on the shell boundary, namely,

$$\frac{1}{H_\eta}\frac{\partial T(\tau, \eta_1, \varphi)}{\partial \eta} = \frac{1}{H_\eta}\cdot\frac{\partial T(\tau, \eta_2, \varphi)}{\partial \eta} = 0. \tag{4}$$

It is assumed that at the beginning ($\tau = 0$) the shell temperature is zero:

$$T(0, \eta, \varphi) = 0. \tag{5}$$

The general case of arbitrary distribution of initial temperatures can be reduced to the analyzed one by the substitution

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* In the case of a spherical shell containing a diathermal (transparent) medium, the problem does not become more difficult if one takes into account the radiative heat exchange between its elements, since a uniform distribution of incoming specific heat fluxes over the inner surface is a distinctive feature of a spherical shell (see, for example, [1]).

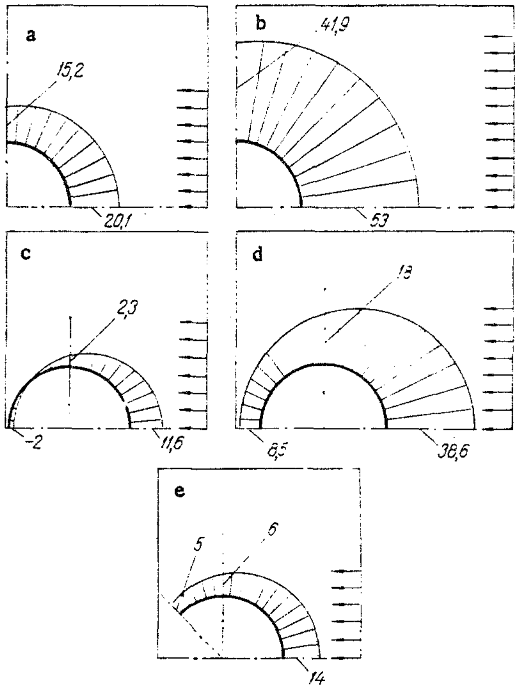


Fig. 3. Distribution of temperatures t along the meridian for the segments of spherical shells for $\varepsilon = 0.9$: a) $\vartheta_1 = \pi/2$, $A_S = 0.4$; b) $\vartheta_1 = \pi/2$, $A_S = 0.9$; c) $\vartheta_1 = \pi$, $A_S = 0.4$; d) $\vartheta_1 = \pi$, $A_S = 0.9$; e) $\vartheta_1 = {}^2/_3\,\pi$, $A_S = 0.4$. The values in the diagrams refer to temperature in °C.

$$T_1 = T - T(0,\ \eta,\ \varphi).$$

The problem (2), (4), (5) can be replaced by an equivalent one by extending it to the entire surface $S + S_1 + S_2$ (Fig. 1) and by introducing fictitious heat fluxes $q_1$ and $q_2$, that is,

$$c\rho\beta H_\xi \frac{\partial T}{\partial \tau} = \lambda\beta H_\xi \Delta_1 T - hT + q + q_1 + q_2, \tag{6}$$

$$q(\tau,\ P) = 0, \quad \text{if} \quad P \in S_1 + S_2,$$

$$q_1(\tau,\ P) = \begin{cases} \sum\limits_{\nu=0}^{\infty} F_{1\nu}^{0}(\tau)\cos(\nu\varphi) + \sum\limits_{\nu=1}^{\infty} F_{1\nu}^{e}(\tau)\sin(\nu\varphi), & P \in S_1, \\ 0 & P \in S + S_2, \end{cases}$$

$$q_2(\tau,\ P) = \begin{cases} \sum\limits_{\nu=0}^{\infty} F_{2\nu}^{0}(\tau)\cos(\nu\varphi) + \sum\limits_{\nu=1}^{\infty} F_{2\nu}^{e}(\tau)\sin(\nu\varphi), & P \in S_2, \\ 0 & P \in S + S_1. \end{cases}$$

The functions $F_{1\nu}^{0}(\tau)$, $F_{1\nu}^{e}(\tau)$, $F_{2\nu}^{0}(\tau)$, $F_{2\nu}^{e}(\tau)$ have to be determined by employing the boundary conditions (4).

If q is an even function of $\varphi$, then $F_{1\nu}^{e}(\tau) = F_{2\nu}^{e}(\tau) = 0$. In this case the solution is sought in the form

$$T = A(\eta) \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} \theta_{k\nu}(\tau) X_{k\nu}(\eta)\cos(\nu\varphi), \tag{7}$$

where $X_{k\nu}(\eta)$ is the orthogonal system of eigenfunctions of the equation

$$[B(\eta) L + N_\nu(\eta)][A(\eta) X_{k\nu}(\eta)] + \mu_k A(\eta) X_{k\nu}(\eta) = 0. \tag{8}$$

The operator L is given by

$$L = \frac{1}{H_\eta^2} \cdot \frac{d^2}{d\eta^2} + \frac{1}{H_\xi H_\eta H_\varphi} \cdot \frac{\partial}{\partial \eta}\left(\frac{H_\xi H_\varphi}{H_\eta}\right)\frac{d}{d\eta} - \frac{\nu^2}{H_\varphi^2}.$$

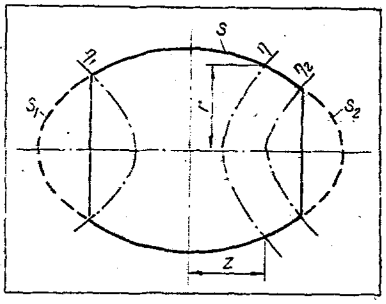


Fig. 4. Spheroidal coordinates.

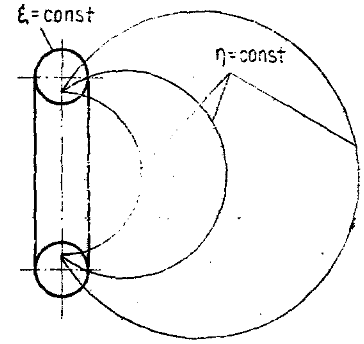


Fig. 5. Toroidal coordinates.

The function $A(\eta)$ is adopted in the same way as for the separation of variables in the Laplace equation [4]. There is no difficulty in selecting the functions $B(\eta)$ and $N_\nu(\eta)$.

In the general case the function q can be represented as a sum of an even ($q^0$) and an odd ($q^e$) function, and the sought solution is equal to the sum of the solutions of Eq. (6) for $q^0$ and $q^e$, respectively.

Substituting the expression (7) in Eq. (6), one obtains the following system of equations:

$$\sum_{k=0}^{\infty} \mu_k X_{k\nu}(\eta)\theta_{k\nu}(\tau) + \left[N_\nu(\eta) + \frac{h}{\lambda\beta H_\xi} B(\eta)\right] \sum_{k=0}^{\infty} X_{k\nu}(\eta)\theta_{k\nu}(\tau)$$
$$+ \frac{B(\eta)}{a} \sum_{k=0}^{\infty} X_{k\nu}(\eta) \frac{d\theta_{k\nu}(\tau)}{d\tau} = \sum_{k=0}^{\infty} X_{k\nu}(\eta) A_{k\nu}(\tau)$$
$$+ F_{1\nu}^0(\tau) \sum_{k=0}^{\infty} e_{k\nu} X_{k\nu}(\eta) + F_{2\nu}^0(\tau) \sum_{k=0}^{\infty} g_{k\nu} X_{k\nu}(\eta), \tag{9}$$

where $A_{k\nu}(\tau)$, $e_{k\nu}$, $g_{k\nu}$ are the expansion coefficients in terms of the eigenfunctions on the surface $S + S_1 + S_2$ for the following expressions:

$$\frac{B(\eta)}{A(\eta)} \cdot \frac{q}{\lambda\beta H_\xi} = \sum_{\nu=0}^{\infty} \sum_{k=0}^{\infty} A_{k\nu}(\tau) X_{k\nu}(\eta) \cos(\nu\varphi);$$
$$\frac{B(\eta)}{A(\eta)} \cdot \frac{q_1}{\lambda\beta H_\xi} = \sum_{\nu=0}^{\infty} F_{1\nu}^0(\tau) \cos(\nu\varphi) \sum_{k=0}^{\infty} e_{k\nu} X_{k\nu}(\eta); \tag{10}$$
$$\frac{B(\eta)}{A(\eta)} \cdot \frac{q_2}{\lambda\beta H_\xi} = \sum_{\nu=0}^{\infty} F_{2\nu}^0(\tau) \cos(\nu\varphi) \sum_{k=0}^{\infty} g_{k\nu} X_{k\nu}(\eta).$$

Of course, the following expansions are valid:

$$\frac{B(\eta) X_{k\nu}(\eta)}{H_\xi} \sum_j c_{jk\nu} X_{j\nu}(\eta); \quad N_\nu(\eta) X_{k\nu}(\eta) = \sum_j d_{jk\nu} X_{j\nu}(\eta). \tag{10'}$$

By employing these expansions, one can replace Eqs. (9) by a system of differential equations for the functions $\theta_{k\nu}(\tau)$:

$$\mu_k \theta_{k\nu}(\tau) + \sum_l d'_{kl\nu} \theta_{l\nu}(\tau) + \frac{1}{a} \sum_l c_{kl\nu} \frac{d\theta_{l\nu}}{d\tau}$$
$$= A_{k\nu}(\tau) + e_{k\nu} F_{1\nu}^0(\tau) + g_{k\nu} F_{2\nu}^0(\tau); \tag{11}$$
$$\left(d'_{kl\nu} = d_{kl\nu} + \frac{h}{\lambda\beta} c_{kl\nu}\right).$$

Formally solving the above equations and substituting the solutions in the boundary conditions (4), one obtains a system of Volterra integral equations of the first kind with difference kernels for the functions $F_{1\nu}^0(\tau)$ and $F_{2\nu}^0(\tau)$:

$$\int_0^\tau K_{1\nu}(\tau-t,\ \eta_1)F^0_{1\nu}(t)\,dt+\int_0^\tau K_{2\nu}(\tau-t,\ \eta_1)F^0_{2\nu}(t)\,dt+M_\nu(\tau,\ \eta_1)=0,$$
$$\int_0^\tau K_{1\nu}(\tau-t,\ \eta_2)F^0_{1\nu}(t)\,dt+\int_0^\tau K_{2\nu}(\tau-t,\ \eta_2)F^0_{2\nu}(t)\,dt+M_\nu(\tau,\ \eta_2)=0. \tag{12}$$

Here $\eta_1$, $\eta_2$ can be regarded as parameters. Equations (12) are integral equations of the convolution type for whose solution it is expedient to use Laplace transforms.

If the surface $S_2$ degenerates into a single point, the system (12) reduces to a single integral equation:

$$\int_0^\tau K_\nu(\tau-t)F^0_\nu(t)\,dt+M_\nu(\tau)=0. \tag{13}$$

Of course, one arrives at equations of the same form also if there are boundary conditions of the first kind when the temperature distribution is specified on the shell boundary.

In the case of shells of constant thickness [the latter corresponds to Eq. (3)] the problem is solved similarly. If q is an even function of $\varphi$, then the solution is sought in the form of (7), where $X_{k\nu}(\eta)$ is an orthogonal system of eigenfunctions for the equation of the form (8) in which the operator L is replaced by the operator $L_1$:

$$L_1=\frac{1}{H_\eta^2}\cdot\frac{d^2}{d\eta^2}+\frac{1}{H_\eta H_\varphi}\cdot\frac{\partial}{\partial\eta}\left(\frac{H_\varphi}{H_\eta}\right)\frac{d}{d\eta}-\frac{\nu^2}{H_\varphi^2}. \tag{14}$$

The functions $\theta_{k\nu}(\tau)$ are the solutions of Eqs. (11) and their coefficients can be obtained by eigenfunction expansions of the expressions similar to (10) and (10') obtained for $\beta H_\zeta = o =$ const.

Several examples are now considered.

Heating a Shell of Constant Thickness Whose Mid-Surface Is a Part of a Sphere. Let $S + S_1 + S_2$ be the surface of a sphere of radius $\xi$. In this case the operator is given by

$$L=\frac{1-\eta^2}{\xi^2}\cdot\frac{d^2}{d\eta^2}-\frac{2\eta}{\xi^2}\cdot\frac{d}{d\eta}-\frac{\nu^2}{\xi^2(1-\eta^2)}.$$

Since the Laplace equation in the spherical coordinate system separates without the aid of an auxiliary function, $A(\eta) = 1$. Moreover, it is obvious that $B(\eta) = \xi^2 =$ const, $N_\nu(\eta) = 0$. For a spherical shell Eq. (8) is of the form

$$(1-\eta^2)\frac{d^2X_{k\nu}}{d\eta^2}-2\eta\frac{dX_{k\nu}}{d\eta}+\left(\mu_k-\frac{\nu^2}{1-\eta^2}\right)X_{k\nu}=0. \tag{15}$$

Thus (7) can be replaced by

$$T=\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}\theta_{k\nu}(\tau)P_k^\nu(\eta)\cos(\nu\varphi);\ \mu_k=k(k+1). \tag{16}$$

Here $P_k^\nu$ are the associated Legendre polynomials.

The system (11) separates for a sphere into individual independent equations [since $B(\eta) =$ const, $N_\nu(\eta) = 0$]:

$$\left[k(k+1)+\frac{h\xi^2}{\lambda\delta}+\frac{\xi^2}{a}\cdot\frac{d}{d\tau}\right]\theta_{k\nu}(\tau)=A_{k\nu}(\tau)+e_{k\nu}F^0_{1\nu}(\tau)+g_{k\nu}F^0_{2\nu}(\tau);$$
$$A_{k\nu}(\tau)=\frac{2k+1}{m\pi}\cdot\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{\xi^2}{\lambda\delta}\int_0^{2\pi}\int_{\eta_1}^{\eta_2}q(\tau,\ \eta,\ \varphi)P_k^\nu(\eta)\cos(\nu\varphi)\,d\eta d\varphi; \tag{17}$$
$$e_{k\nu}=\frac{2k+1}{2}\cdot\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{\xi^2}{\lambda\delta}\int_{-1}^{\eta_1}P_k^\nu(\eta)\,d\eta;\ g_{k\nu}$$
$$=\frac{2k+1}{2}\cdot\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{\xi^2}{\lambda\delta}\int_{\eta_2}^{1}P_k^\nu(\eta)\,d\eta;$$

$\nu = 0, 1, 2, \ldots$; for $\nu = 0, m = 4$; for $\nu \neq 0, m = 2$. Applying the Laplace transform with respect to $\tau$ to Eqs. (17) and taking into account that $\theta(0) = 0$, by virtue of (5) one obtains the following for the transform $\vartheta_{k\nu}(s)$ of the function $\theta_{k\nu}(\tau)$:

$$\vartheta_{k\nu}(s) = \frac{a}{\xi^2} \cdot \frac{a_{k\nu}(s) + e_{k\nu} f_{1\nu}(s) + g_{k\nu} f_{2\nu}(s)}{s + \gamma_k};$$

$$\gamma_k = \frac{h\xi^2}{c\rho\delta} + k(k+1)\frac{a}{\xi^2}.$$

Thus, the transform $t(s, \eta, \varphi)$ of the function $T(\tau, \eta, \varphi)$ is

$$t(s, \eta, \varphi) = \frac{a}{\xi^2} \sum_{\nu=0}^{\infty} \cos(\nu\varphi) \left[ \sum_{k=\nu}^{\infty} \frac{a_{k\nu} P_k^\nu(\eta)}{s + \gamma_k} + f_{1\nu}(s) \sum_{k=0}^{\infty} \frac{e_{k\nu} P_k^\nu(\eta)}{s + \gamma_k} + f_{2\nu}(s) \sum_{k=\nu}^{\infty} \frac{g_{k\nu} P_k^\nu(\eta)}{s + \gamma_k} \right]. \tag{18}$$

In the general case ($\eta_2 \neq 0$) the transforms $f_{1\nu}(s)$ and $f_{2\nu}(s)$ are found from the conditions

$$\frac{\partial t(s, \eta_1, \varphi)}{\partial \eta} = 0, \quad \frac{\partial t(s, \eta_2, \varphi)}{\partial \eta} = 0 \tag{19}$$

and are given, respectively, by

$$f_{1\nu}(s) = \frac{k_{2\nu}(s, \eta_1) m_\nu(s, \eta_2) - k_{2\nu}(s, \eta_2) m_\nu(s, \eta_1)}{k_{1\nu}(s, \eta_1) k_{2\nu}(s, \eta_2) - k_{1\nu}(s, \eta_2) k_{2\nu}(s, \eta_1)},$$

$$f_{2\nu}(s) = \frac{k_{1\nu}(s, \eta_2) m_\nu(s, \eta_1) - k_{1\nu}(s, \eta_1) m_\nu(s, \eta_2)}{k_{1\nu}(s, \eta_1) k_{2\nu}(s, \eta_2) - k_{1\nu}(s, \eta_2) k_{2\nu}(s, \eta_1)},$$

$$k_{1\nu}(s, \eta) = \sum_{k=\nu}^{\infty} \frac{e_{k\nu} (P_k^\nu)'(\eta)}{s + \gamma_k}, \quad k_{2\nu}(s, \eta) = \sum_{k=\nu}^{\infty} \frac{g_{k\nu} (P_k^\nu)'(\eta)}{s + \gamma_k},$$

$$m_\nu(s, \eta) = \sum_{k=\nu}^{\infty} \frac{a_{k\nu}(s) (P_k^\nu)'(\eta)}{s + \gamma_k}, \quad (P_k^\nu)'(\eta) = \frac{dP_k^\nu(\eta)}{d\eta}.$$

Going back to the original functions, one obtains by employing the expansion and convolution theorems (of Borel) [5] the temperature distribution given by

$$T = \frac{1}{c\rho\delta} \sum_{\nu=0}^{\infty} \cos(\nu\varphi) \int_0^\tau \int_0^{2\pi} \int_{\eta_1}^{\eta_2} q(\tau', y, \Phi) \Psi_\nu(\tau - \tau', y, \eta) \cos(\nu\Phi)\, dy\, d\Phi\, d\tau', \tag{20}$$

$$\Psi_0(\tau, y, \eta) = c_0(y) \exp(-\gamma_0 \tau) + \sum_{l=1}^{\infty} c_{l0}(y, \eta) \exp(-\alpha_{l0}\tau),$$

$$\Psi_\nu(\tau, y, \eta) = \sum_{l=1}^{\infty} c_{l\nu}(y, \eta) \exp(-\alpha_{l\nu}\tau),$$

$\alpha_{l\nu}$ being the roots of the equations $\chi_\nu(-\alpha_{l\nu}) = 0$,

$$\chi_\nu(s) = \sum_{k=\nu}^{\infty} \frac{e_{k\nu} (P_k^\nu)'(\eta_1)}{s + \gamma_k} \sum_{k=\nu}^{\infty} \frac{g_{k\nu} (P_k^\nu)'(\eta_2)}{s + \gamma_k} - \sum_{k=\nu}^{\infty} \frac{e_{k\nu} (P_k^\nu)'(\eta_2)}{s + \gamma_k} \sum_{k=\nu}^{\infty} \frac{g_{k\nu} (P_k^\nu)'(\eta_1)}{s + \gamma_k},$$

$$c_0(y) = \frac{1}{4\pi} \left[ 1 - e_{00} \sum_{k=1}^{\infty} (2k+1) \frac{P_k(y) (P_k)'(\eta_1)}{\gamma_k - \gamma_0} \Big/ \sum_{k=1}^{\infty} \frac{e_{k0} (P_k)'(\eta_1)}{\gamma_k - \gamma_0} \right],$$

$$c_{l\nu}(y, \eta) = \frac{1}{m\pi} \left[ \frac{\omega_{1\nu}(y, -\alpha_{l\nu})}{\chi_\nu'(-\alpha_{l\nu})} \sum_{k=\nu}^{\infty} \frac{e_{k\nu} P_k^\nu(\eta)}{\gamma_k - \alpha_{l\nu}} \right.$$

$$+\frac{\omega_{2\nu}(y,-\alpha_{l\nu})}{\chi'_\nu(-\alpha_{l\nu})}\sum_{k=\nu}^{\infty}\frac{g_{k\nu}P_k^\nu(\eta)}{\gamma_k-\alpha_{l\nu}}\Bigg],$$

$$\omega_{1\nu}(y,s)=k_{2\nu}(s,\eta_1)\sum_{k=\nu}^{\infty}(2k+1)\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{P_k^\nu(y)(P_k^\nu)'(\eta_2)}{s-\gamma_k}$$

$$-k_{2\nu}(s,\eta_2)\sum_{k=\nu}^{\infty}(2k+1)\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{(P_k^\nu)'(\eta_1)P_k^\nu(y)}{s+\gamma_k},$$

$$\omega_{2\nu}(y,s)=k_{1\nu}(s,\eta_2)\sum_{k=\nu}^{\infty}(2k-1)\frac{(k-\nu)!}{(k-\nu)!}\cdot\frac{P_k^\nu(y)(P_k^\nu)'(\eta_1)}{s-\gamma_k}$$

$$-k_{1\nu}(s,\eta_1)\sum_{k=\nu}^{\infty}(2k+1)\frac{(k-\nu)!}{(k+\nu)!}\cdot\frac{P_k^\nu(y)(P_k^\nu)'(\eta_2)}{s-\gamma_k}.$$

If the surface $S_2$ degenerates into a point ($\eta_2 = 1$), then Eq. (20) is much simpler, since then $g_{k\nu} = 0$.

In Fig. 2 the results of calculations of heating segments of a spherical shell which correspond to the angles $\vartheta = \arccos\eta_1 = \pi/2;\ 2\pi/3;\ \pi$ for $\varepsilon = 0.9$ and $A_s = 0.4;\ 0.9$ are shown. It was assumed that the elements of these segments are subjected to the flow of solar radiation $q_s = A_s E\eta H(\eta)$ [$H(\eta) = 1$ for $\eta > 0$ and 0 for $\eta < 0$]. Moreover, it was assumed in the calculations that the heat exchange between the shell and the medium (which is either inside or outside the shell and has the temperature $T_1 = 303°K$) is described by the heat-emission coefficient $h_1 = 5.8\ W/m^2\cdot deg$. Thus, the specific heat flux appearing in the equations is

$$q=q_s+h_1(T_1-T_{in})-\varepsilon\sigma(4T_0^3T_{in}-3T_0^4);\ h=h_1-4\varepsilon\sigma T_0.$$

Here $T_{in}$ and $T_0$ are the initial temperature of the shell and the mean value of the temperature for the entire duration of heating. The following initial values were used in the calculations: $\xi = 0.1$ m; $\delta = 0.0015$ m; $\rho = 2700\ kg/m^3$; $\lambda = 102.3\ W/m\cdot deg$ (alloy Al + Mg); $a = 0.152\ m^2/h$; $E = 1396\ W/m^2$; and $T_{in} = 293°K$.

It is an interesting feature that the temperature does not change monotonically at the front point ($\vartheta = \arccos\eta = 0$) for $A_s = 0.4$, this being due to the fact that at the initial stage the heating of the corresponding shell element takes place more rapidly than its cooling due to the flow of heat to the rear region, the latter becomes cooler due to emission into space. For the states under consideration the steady-state temperature distribution along a meridian is shown in Fig. 3a-d.

Heating of Shell Bounded by Two Close Prolate Cofocal Spheroidal Surfaces. Let $S + S_1 + S_2$ be the surface of a prolate spheroid. Prolate spheroidal coordinates are formed by rotating the elliptic coordinate about the longer axis of the ellipse [4]. The foci of the spheroid are at the points $r = 0$, $z = \pm b$ (Fig. 4). Consequently, one has [4]

$$z=b\eta\xi,\ r=b\sqrt{(\xi^2-1)(1-\eta^2)},\ 1\leqslant\xi<\infty,\ -1\leqslant\eta\leqslant1,\ 0\leqslant\varphi\leqslant2\pi,$$

$$L=\frac{1}{b^2(\xi^2-\eta^2)}\left[(1-\eta^2)\frac{d^2}{d\eta^2}-2\eta\frac{d}{d\eta}-\frac{\nu^2}{1-\eta^2}-\frac{\nu^2}{\xi^2-1}\right].$$

In the adopted coordinate system one has

$$A(\eta)=1,\quad B(\eta)=b^2(\xi^2-\eta^2),\ N_\nu(\eta)=\frac{\nu^2}{\xi^2-1}B(\eta),$$

and the functions $X_{k\nu}(\eta)$ satisfy Eq. (15); the temperature distribution must, therefore, be sought in the form (16). Having applied the Laplace transform, Eqs. (9) in this case assume the form

$$\sum_{k=\nu}^{\infty}\Bigg\{\Bigg[k(k+1)\vartheta_{k\nu}(s)+b^2(\xi^2-\eta^2)\left(\frac{h}{\lambda\beta}\cdot\frac{1}{H_\xi}+\frac{\nu^2}{\xi^2-1}\right)\vartheta_{k\nu}(s)$$

$$+\frac{b^2}{a}(\xi^2-\eta^2)s\vartheta_{k\nu}(s)\Bigg]P_k^\nu(\eta)-[a_{k\nu}(s)+e_{k\nu}f_{1\nu}(s)+g_{k\nu}f_{2\nu}(s)]P_k^\nu(\eta)\Bigg\}=0.$$

By expanding the functions $\eta^2P_k^\nu(\eta)$ and $\sqrt{\xi^2-\eta^2}P_k^\nu(\eta)$ into a series of associated Legendre polynomials $P_k^\nu(\eta)$:

$$\eta^2P_k^\nu(\eta)=\sum_j d_{jk\nu}P_j^\nu(\eta),\ \sqrt{\xi^2-\eta^2}\,P_k^\nu(\eta)=\sum_j d_{jk\nu}^0P_j^\nu(\eta),$$

$$d_{jk\nu}=\frac{2j+1}{2}\cdot\frac{(k-\nu)!}{(k+\nu)!}\int_{-1}^{1}\eta^2P_k^\nu(\eta)P_j^\nu(\eta)\,d\eta,$$

$$d^0_{jk\nu} = \frac{2j+1}{2} \cdot \frac{(k-\nu)!}{(k+\nu)!} \int_{-1}^{1} \sqrt{\xi^2 - \eta^2}\, P^\nu_k(\eta)\, P^\nu_j(\eta)\, d\eta,$$

one obtains a system of equations for the transforms $\vartheta_{k\nu}(s)$ of the functions $\theta_{k\nu}(\tau)$

$$\left[k(k+1) + b^2\xi^2\left(\frac{s}{a} + \frac{\nu^2}{\xi^2 - 1}\right)\right] \vartheta_{k\nu}(s) = b^2\left(\frac{s}{a} + \frac{\nu^2}{\xi^2 - 1}\right)$$

$$\times \sum_l d_{kl\nu}\vartheta_{l\nu}(s) + b\sqrt{\xi^2 - 1}\,\frac{h}{\lambda\beta} \sum_l d^0_{kl\nu}\vartheta_{l\nu}(s) + a_{k\nu}(s) + e_{k\nu}f_{1\nu}(s) + g_{k\nu}f_{2\nu}(s). \tag{21}$$

Under certain conditions the infinite system of linear equations (21) can be solved by the method of successive approximations. We shall dwell at some length on how the components of the temperature field which possess cylindrical symmetry ($\nu = 0$) should be determined in the case of a moderate eccentricity of the spheroid such that $\eta^4/\xi^4 \ll 1$; consequently,

$$\sqrt{\xi^2 - \eta^2} \approx \xi\left(1 - \frac{1}{2}\frac{\eta^2}{\xi^2}\right),$$

$$\sqrt{\xi^2 - \eta^2}\, P^\nu_k(\eta) \approx \xi\left[P^\nu_k(\eta) - \frac{1}{2\xi^2}\sum_j d_{jk\nu}P^\nu_j(\eta)\right].$$

Then the system of equations (21) can be rewritten as

$$\vartheta_{k0}(s) = \frac{1}{\xi^2} \cdot \frac{s + \gamma_0/2}{s + \gamma_k}\,[c_{k,k-2}\vartheta_{k-2,0}(s) + c_{kk}\vartheta_{k0}(s) + c_{k,k+2}\vartheta_{k+2,0}(s)]\,\frac{au_{k0}(s)}{b^2\xi^2(s + \gamma_k)}, \tag{22}$$

$$k = 0,\ 1,\ 2,\ \ldots,$$

where

$$\gamma_k = \frac{h}{c\rho\beta b} \cdot \frac{\sqrt{\xi^2 - 1}}{\xi} + k(k+1)\frac{a}{b^2\xi^2}, \quad u_{k0}(s) = a_{k0}(s) + e_{k0}f_{10}(s) + g_{k0}f_{20}(s),$$

$$c_{kl} = d_{kl0} = \frac{2k+1}{2}\int_{-1}^{1} \eta^2 P_k(\eta)\, P_l(\eta)\, d\eta.$$

Since $c_{kj} \nless 0$, the system (2) is completely regular [6] if

$$\frac{1}{\xi^2}\left|\frac{s + \gamma_0/2}{s + \gamma_k}\right| \sigma_k \leqslant \varepsilon < 1; \quad \sigma_k = \sum_j c_{kj}, \tag{23}$$

but Re $s > 0$; $\gamma_k > \gamma_0 > 0$; Im $\gamma_k = 0$. Consequently, one has $|(s + \gamma_0/2)/(s + \gamma_k)| \leq 1$ and the condition (23) is equivalent to the condition

$$\xi^2 > \max \sigma_k.$$

Taking into account that

$$c_{k,k-2} = \frac{k(k-1)}{(2k-1)(2k-3)}, \quad c_{kk} = \frac{(k+1)^2}{(2k+1)(2k+3)} + \frac{k^2}{(2k+1)(2k-1)},$$

$$c_{k,k+2} = \frac{(k+1)(k+2)}{(2k+3)(3k+5)}, \quad c_{k,k\pm j} = 0 \text{ for } j \neq 0;\ 2,$$

we calculate the sum of the coefficients

$$\sigma_k = \frac{32k^5 + 80k^4 - 48k^3 - 152k^2 - 14k + 21}{32k^5 + 80k^4 - 80k^3 - 200k^2 + 18k + 45} \quad (k = 2,\ 3,\ 4,\ \ldots),$$

$$\sigma_0 = 7/15,\ \sigma_1 = 33/35,\ \max\sigma_k = \sigma_2 = 29/21 \approx 1.38.$$

Thus, for $\xi^2 > 29/21$ if $u_{k0} \leq \Omega < \infty$, then the system (22) has only one bounded solution which can be found by using successive approximations starting with any bounded system of initial values [6].

The terms $c_{kk}\vartheta_{k0}(s)$ in Eqs. (22) are now transferred to the left-hand side. The condition of complete regularity is not infringed by it:

$$\vartheta_{k0}(s) = \frac{s + \gamma_0/2}{(\xi^2 - c_{kk})(s + \zeta_k)} [c_{k,k-2}\vartheta_{k-2,0}(s) + c_{k,k+2}\vartheta_{k+2,0}(s)] + \frac{au_{k0}}{b^2(\xi^2 - c_{kk})(s + \zeta_k)}, \tag{24}$$

where $\zeta_k = \dfrac{\xi^2\gamma_k - c_{kk}\gamma_0/2}{\xi^2 - c_{kk}}$ $(k = 0, 1, 2, \ldots)$.

The system of equations (24) is now solved by successive approximations in which $\vartheta_{k0}(s) = 0$ is adopted as its initial value. After two iterations one finds

$$\vartheta_{k0}(s) = \frac{au_{k0}(s)}{b^2(\xi^2 - c_{kk})(s + \zeta_k)} + \frac{ac_{k,k-2}(s + \gamma_0/2)u_{k-2,0}(s)}{b^2(\xi^2 - c_{kk})(\xi^2 - c_{k-2,k-2})(s + \zeta_k)(s + \zeta_{k-2})} + \frac{ac_{k,k+2}(s + \gamma_0/2)u_{k+2,0}(s)}{b^2(\xi^2 - c_{kk})(\xi^2 - c_{k+2,k+2})(s + \zeta_k)(s + \zeta_{k+2})}; \ (k = 0, 1, 2, \ldots). \tag{25}$$

The convergence rate of the procedure improves with the parameter $\xi$ increasing. Returning to the originals, one obtains the following for $\eta_1 = -1$, $\eta_2 = 1$ $(u_{k0} = \alpha_{k0})$:

$$\theta_{k0}(\tau) = \frac{a}{b^2}\left[\frac{A_{k0}(\tau) * \exp(-\zeta_k\tau)}{\xi^2 - c_{kk}} + \frac{c_{k,k-2}A_{k-2,0}(\tau)}{(\xi^2 - c_{kk})(\xi^2 - c_{k-2,k-2})} * \frac{(\gamma_0/2 - \zeta_k)\exp(-\zeta_k\tau) - (\gamma_0/2 - \zeta_{k-2})\exp(-\zeta_{k-2}\tau)}{\zeta_{k-2} - \zeta_k} + \frac{c_{k,k+2}A_{k+2,0}(\tau)}{(\xi^2 - c_{kk})(\xi^2 - c_{k+2,k+2})} * \frac{(\gamma_0/2 - \zeta_k)\exp(-\zeta_k\tau) - (\gamma_0/2 - \zeta_{k+2})\exp(-\zeta_{k+2}\tau)}{\zeta_{k+2} - \zeta_k}\right]$$

$$(k = 0, 1, 2, \ldots).$$

Here

$$\Phi_1(\tau) * \Phi_2(\tau) = \int_0^\tau \Phi_1(\tau - t)\,\Phi_2(t)\,dt$$

is the convolution of two functions $\Phi_1(\tau)$ and $\Phi_2(\tau)$. In the general case (for $\eta_1 \neq -1$; $\eta_2 \neq 1$) the functions $f_{1\nu}(s)$, $f_{2\nu}(s)$ are obtained by substituting the expressions (25) and similar expressions for $\nu \neq 0$ into the boundary conditions (20). One can then find the temperature distribution.

Similarly, by using oblate spheroidal coordinates [4], one can obtain the temperature distribution when a shell bounded by two close oblate cofocal spheroidal surfaces is heated up.

Heating a Shell of Constant Thickness Whose Mid-Surface Is a Portion of Torus. Let $S + S_1 + S_2$ be the torus surface. Toroidal coordinates are obtained by revolving the bipolar coordinate system around the perpendicular passing through the middle of the straight line joining the poles (Fig. 5).

Therefore,

$$z = \frac{b \sin\eta}{\operatorname{ch}\xi - \cos\eta}; \quad r = \frac{b \operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\eta};$$

$$0 \leqslant \xi < \infty; \quad 0 \leqslant \eta \leqslant 2\pi; \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Since shells of constant thickness ($\delta$ = const) are now analyzed, to determine the functions $X_{k\nu}(\eta)$ in Eq. (8) one replaces the operator L by the operator $L_1$ of (14):

$$L_1 = \frac{(\operatorname{ch}\xi - \cos\eta)^2}{b^2}\left(\frac{d^2}{d\eta^2} - \frac{\nu^2}{\operatorname{sh}^2\xi}\right).$$

Thus,

$$A(\eta)=1;\quad B(\eta)=\frac{b^2}{(\operatorname{ch}\xi-\cos\eta)^2};\ N_\nu(\eta)=\frac{\nu^2}{\operatorname{sh}^2\xi},$$

and Eq. (8) reduces in this case to the following equation:

$$\frac{d^2X_{k\nu}(\eta)}{d\eta^2}+\mu_k X_{k\nu}=0.$$

Then

$$\mu_k=k^2;\quad X'_{k\nu}=\cos(k\eta);\quad X''_{k\nu}=\sin(k\eta)$$

and the temperature distribution in the shell is now sought in the form

$$T=\sum_{\nu=0}^{\infty}\left[\sum_{k=0}^{\infty}\theta'_{k\nu}(\tau)\cos(k\eta)+\sum_{k=1}^{\infty}\theta''_{k\nu}(\tau)\sin(k\eta)\right]\cos(\nu\varphi). \tag{26}$$

In this case Eqs. (9) become

$$\sum_{k=0}^{\infty}k^2\cos(k\eta)\,\theta'_{k\nu}(\tau)+\left[\frac{\nu^2}{\operatorname{sh}^2\xi}+\frac{h}{\lambda\delta}\cdot\frac{b^2}{(\operatorname{sh}\xi-\cos\eta)^2}\right]$$
$$\times\sum_{k=0}^{\infty}\cos(k\eta)\,\theta'_{k\nu}(\tau)+\frac{b^2}{a(\operatorname{ch}\xi-\cos\eta)}$$
$$\times\sum_{k=0}^{\infty}\cos(k\eta)\,\frac{d\theta'_{k\nu}(\tau)}{d\tau}-\sum_{k=0}^{\infty}\cos(k\eta)\,A'_{k\nu}(\tau)$$
$$-F^0_{1\nu}(\tau)\sum_{k=0}^{\infty}e'_{k\nu}\cos(k\eta)-F^0_{2\nu}(\tau)\sum_{k=0}^{\infty}g'_{k\nu}\cos(k\eta)=0. \tag{27}$$

In a similar manner one can write down the equations for $\theta''_{k\nu}(\tau)$ corresponding to the eigenfunctions $X''_{k\nu}$.

By finding the Fourier expansions of the expressions

$$\frac{\cos(k\eta)}{(\operatorname{ch}\xi-\cos\eta)^2}=\sum_j c'_{jk}\cos(j\eta),\qquad \frac{\sin(k\eta)}{(\operatorname{ch}\xi-\cos\eta)^2}=\sum_j c''_{jk}\sin(j\eta),$$

one obtains two independent systems of equations for the transforms $\vartheta'_{k\nu}$, $\vartheta''_{k\nu}$ of the functions $\theta'_{k\nu}$, $\theta''_{k\nu}$:

$$(s+\zeta'_{k\nu})\,\vartheta'_{k\nu}(s)=-\sum_{l\neq k}\frac{c'_{kl}}{c'_{kk}}\left(s+\frac{a}{b^2}\cdot\frac{hb^2}{\lambda\delta}\right)\vartheta'_{l\nu}+\frac{a}{b^2}\cdot\frac{u'_{k\nu}}{c'_{kk}},$$
$$(s+\zeta''_{k\nu})\,\vartheta''_{k\nu}(s)=-\sum_{l\neq k}\frac{c''_{kl}}{c''_{kk}}\left(s+\frac{a}{b^2}\cdot\frac{hb^2}{\lambda\delta}\right)\vartheta''_{l\nu}+\frac{a}{b^2}\cdot\frac{u''_{k\nu}}{c''_{kk}}. \tag{28}$$

Here

$$\zeta'_{k\nu}=\frac{ah}{\lambda\delta}+\frac{a}{b^2c'_{kk}}\left(k^2+\frac{\nu^2}{\operatorname{sh}^2\xi}\right),$$
$$\zeta''_{k\nu}=\frac{ah}{\lambda\delta}+\frac{a}{b^2c''_{kk}}\left(k^2+\frac{\nu^2}{\operatorname{sh}^2\xi}\right).$$

By introducing another variable $h'_{k\nu}(s)=(s+\zeta'_{k\nu})\,\vartheta'_{k\nu}(s)$, one can rewrite the first system of the equations (28) as

$$h'_{k\nu}(s)=-\sum_{l\neq k}\frac{c'_{kl}}{c'_{kk}}\cdot\frac{s+\varkappa}{s+\zeta'_{l\nu}}\,h'_{l\nu}(s)+\frac{a}{b^2}\cdot\frac{u'_{k\nu}(s)}{c'_{kk}}, \tag{29}$$

where $\varkappa=ah/\lambda\delta$.

In the approximate solution we shall confine our considerations to a finite interval by integrating over s in the inverse Laplace transform, that is, one assumes that $|s|\le\sigma<\infty$. Then

$$\left|\frac{s+\varkappa}{s+\zeta'_{l\nu}}\right| = O(l^{-2}).$$

Moreover, by taking into account that $c'_{kk} = O\left(\frac{1}{\cosh^2\xi}\right)$, $c'_{kl} = O\left(\frac{1}{\cosh^3\xi}\right)$ $(k \neq l)$, one obtains the following estimate:

$$\sum_{l \neq k}\left|\frac{c'_{kl}}{c'_{kk}}\,\frac{s-\varkappa}{s+\zeta'_{l\nu}}\right| = O\left(\frac{1}{\cosh\xi}\right).$$

Thus, for sufficiently high values of the parameter $\xi$ and with $|s| \leq \sigma < \infty$ the system (29) is completely regular and can be solved by successive approximations starting with any bounded system of initial values provided $u'_{k\nu}(s) \leq u < \infty$. Adopting $h'_{k\nu}(s) = 0$ for the initial values one find after two iterations

$$\vartheta'_{k\nu}(s) = \frac{a}{c'_{kk}b^2}\left[\frac{u'_{k\nu}(s)}{s+\zeta'_{k\nu}} - \sum_{l \neq k}\frac{c'_{kl}}{c'_{ll}}\,\frac{s-\varkappa}{(s+\zeta'_{k\nu})(s+\zeta'_{l\nu})}\,u'_{l\nu}(s)\right] \tag{30}$$

(for higher $\xi$ the convergence of the iterative procedure is improved). Going back to the originals in the above expression with $\eta_1 = 0$, $\eta_2 = 2\pi$ $[u_{k\nu}(s) = a'_{k\nu}(s)]$, one obtains

$$\theta'_{k\nu} = \frac{a}{b^2c'_{kk}}\Bigg[A'_{k\nu}(\tau) * \exp(-\zeta_{k\nu}\tau) - \sum_{l=k}\frac{c'_{kl}}{c'_{ll}}A'_{l\nu}(\tau) \cdot \frac{(\zeta'_{l\nu}-\varkappa)\exp(-\zeta'_{l\nu}\tau) - (\zeta'_{k\nu}-\varkappa)\exp(-\zeta'_{k\nu}\tau)}{\zeta'_{l\nu}-\zeta'_{k\nu}}\Bigg].$$

In the latter formula one should consider only the first n terms of the sum so that $|\zeta'_{n\nu}| < \sigma$. In the general case (for $\eta_1 \neq 0$, $\eta_2 \neq 2\pi$) the functions $f_{1\nu}(s)$ and $f_{2\nu}(s)$ are found by substituting the expressions (28) into the boundary conditions (20).

## NOTATION

T, temperature; $T_1$, temperature of the medium; $T_{in}$, initial temperature of the shell; c, $\rho$, $\lambda$, specific heat, density, thermal-conductivity coefficient of the shell material, respectively; h, effective coefficient of heat-emission from the shell; $h_1$, coefficient of heat-emission to the medium either inside or outside the shell; $a$, thermal diffusivity; $\sigma$, Stefan–Boltzmann constant; $\delta$, shell thickness; L, characteristic dimension along the shell; $\xi$, $\eta$, $\varphi$, curvilinear coordinate system; $H_\xi$, $H_\eta$, $H_\varphi$, corresponding Lamé coefficients; $\beta = \delta/H_\xi$; S, $S_1$, $S_2$, surface areas of $\xi$ = const corresponding to shell mid-surface and its complements to a complete surface coordinate; q, specific heat flux on shell; $q_1$, $q_2$, specific fictitious heat fluxes in regions $S_1$ and $S_2$; $X_{k\nu}(\eta)$ complete orthogonal system of eigenfunctions of Eq. (8) regular on that interval of values of argument $\eta$ which corresponds to the entire coordinate surface $\xi$ = const; $A(\eta)$, multiplier introduced for the separation of variables; $F^0_j(\tau)$, $F^e_j(\tau)$, coefficients of Fourier series expansion of the functions $q_j$ (j = 1, 2); the superscripts 0, e correspond to even or odd functions of $\varphi$; $P^\nu_k(\eta)$, associated Legendre polynomials; $\vartheta_{k\nu}(s)$, $a_{k\nu}(s)$, $f_{1\nu}(s)$, $f_{2\nu}(s)$, transforms of the functions $\theta_{k\nu}(\tau)$, $A_{k\nu}(\tau)$, $F^0_{1\nu}(\tau)$, $F^0_{2\nu}(\tau)$.

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